Algebraic geometry 1

Exercise Sheet 10

PD Dr. Maksim Zhykhovich Winter Semester 2025, 09.01.2026

Exercise 1. Let X be a Noetherian topological space and write $X = X_1 \cup \cdots \cup X_r$, where X_i are irreducible components of X. Show that

$$\dim X = \max\{\dim X_i \mid i = 1, \ldots, r\}.$$

Exercise 2. Let X be a projective variety (irreducible projective algebraic set) and let $U \subset X$ be a non-empty open subset. Show that $\dim U = \dim X$.
Hint: Use Exercise 4.1, Sheet 9, and Proposition 6.9.

Exercise 3. (1) Let $X \subset \mathbb{P}^n$ be a projective algebraic set. Show that X is a hypersurface if and only if X is a zero set of a single homogeneous polynomial F of positive degree.
Hint: In the lecture we proved such a statement for hypersurfaces in $\mathbb{A}^n$ (see Corollary 6.6 and Proposition 6.8).
(2) Show that any two curves in $\mathbb{P}^2$ have a non-empty intersection.

Exercise 4. Let A be a factorial ring and let ρ be a prime ideal of height 1 (that is the zero ideal is the only prime ideal strictly contained in ρ). Show that ρ is principal.
Remark: We used this exercise in the proof of Proposition 6.8.

Exercise 5. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $F \in K[X_1, \ldots, X_n]$ be a polynomial that does not vanish identically on X. Assume that $X \cap V(F) \neq \emptyset$. Show that
(1) $\dim X \cap V(F) = \dim X - 1$
(2) Every irreducible component of $X \cap V(F)$ has dimension $\dim X - 1$.